NORTH SYDNEY BOYS HIGH SCHOOL

100 Years 1912 ~ 2012

# MATHEMATICS (EXTENSION 1)

2012 HSC Course Assessment Task 2

March 12, 2012

**General instructions**

- Working time – 55 minutes. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

**SECTION I**

- Mark your answers on the answer sheet provided (numbered as page 5)

**SECTION II**

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

**STUDENT NUMBER:** .......................... **# BOOKLETS USED:** .....

**Class** (please ✔)

◯ 12M4A – Mr Weiss
◯ 12M4B – Mr Ireland
◯ 12M4C – Mr Fletcher
◯ 12M3C – Ms Ziaziaris
◯ 12M3D – Mr Lowe
◯ 12M3E – Mr Lam

Marker's use only.

| QUESTION | 1-6 | 7 | 8 | 9 | 10 | 11 | 12 | Total | % |
|---|---|---|---|---|---|---|---|---|---|
| MARKS | $\overline{6}$ | $\overline{10}$ | $\overline{10}$ | $\overline{8}$ | $\overline{8}$ | $\overline{4}$ | $\overline{4}$ | $\overline{50}$ | |

## Section I: Objective response

Mark your answers on the multiple choice sheet provided. **Marks**

**1.** If $y = e^{\ln x}$, then $\dfrac{dy}{dx} =$ **1**

(A) 1 (C) $\frac{1}{x}$

(B) 0 (D) none of these

**2.** If $\log_a b = c$, then **1**

(A) $a = b^c$ (C) $b = c^a$

(B) $c = b^a$ (D) $b = a^c$

**3.** A logarithm is another name for **1**

(A) a base (C) an operator

(B) an index (D) none of these

**4.** If $x^{k+3} = e^{7\ln x}$, the value of $k$ is **1**

(A) $e$ (C) $-3$

(B) 0 (D) 4

**5.** $\displaystyle\int \frac{e^x}{1+e^x}\,dx$ evaluates to **1**

(A) $e^x + C$ (C) $\ln(1+e^x) + C$

(B) $\ln\left(1+e^{2x}\right) + C$ (D) $\ln(1-e^x) + C$

**6.** Given that $\log_8 2 = \log_x 5$, then $x =$ **1**

(A) $\frac{1}{2}$ (C) 2

(B) $\frac{1}{3}$ (D) none of the above

**Examination continues overleaf...**

## Section II: Short answer

**Question 7** (10 Marks) Commence a NEW page. **Marks**

(a) Draw an accurate sketch of the function $y = \log_e(2x+1)$, showing its essential features. State its domain and range. **4**

(b) Differentiate:

i. $\log_e\left(\dfrac{2x+1}{3x-7}\right)$. **2**

ii. $5^x$. **2**

iii. $x^3e^{-3x}$. **2**

**Question 8** (10 Marks) Commence a NEW page. **Marks**

(a) Write the primitive of

i. $\dfrac{x}{9+x^2}$ **2**

ii. $\dfrac{2}{x} + 5e^x$. **2**

iii. $\dfrac{6-2x-x^2}{x}$. **2**

iv. $x\left(x^2+4\right)^5$. **2**

(b) Evaluate $\displaystyle\int_1^4 y\,dx$ if $xy = 1$. **2**

**Question 9** (8 Marks) Commence a NEW page. **Marks**

(a) i. Show that $\dfrac{d}{dx}(x\ln x - x) = \ln x$. **2**

ii. Hence or otherwise, find $\displaystyle\int \ln x^2\,dx$. **2**

(b) For the curve $y = \dfrac{e^x}{x^2+1}$, find the stationary point and determine its nature. **4**

**Question 10** (8 Marks) Commence a NEW page. **Marks**

(a) The gradient of a curve is given by $\frac{dy}{dx} = \frac{2}{2x-1}$ and the curve passes through $(1, \log_e 3)$. Find the equation of the curve. **3**

(b) i. Evaluate $\int_0^4 \left(x^2 - 2x\right)\, dx$. **2**

ii. Find the area bounded by the curve $y = x^2 - 2x$, the $x$ axis and the ordinates $x = 0$ and $x = 4$. **2**

iii. What do you notice about your answers to parts (i) and (ii)? Give a brief explanation. **1**

**Question 11** (4 Marks) Commence a NEW page. **Marks**

(i) Find the volume formed when the portion of the curve $xy = 1$ between $x = 1$ and $x = a$ is rotated about the $x$ axis. **3**

(ii) What is the limiting value of this volume as $a \to \infty$? **1**

**Question 12** (4 Marks) Commence a NEW page. **Marks**

Prove by mathematical induction: **4**

$$1 \times 2 + 3 \times 4 + 5 \times 6 + \cdots + 2n(2n-1) = \frac{n(n+1)(4n-1)}{3}$$

**End of paper.**

## Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

**STUDENT NUMBER:** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

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**1** – Ⓐ Ⓑ Ⓒ Ⓓ
**2** – Ⓐ Ⓑ Ⓒ Ⓓ
**3** – Ⓐ Ⓑ Ⓒ Ⓓ
**4** – Ⓐ Ⓑ Ⓒ Ⓓ
**5** – Ⓐ Ⓑ Ⓒ Ⓓ
**6** – Ⓐ Ⓑ Ⓒ Ⓓ

# Suggested Solutions

## Section I

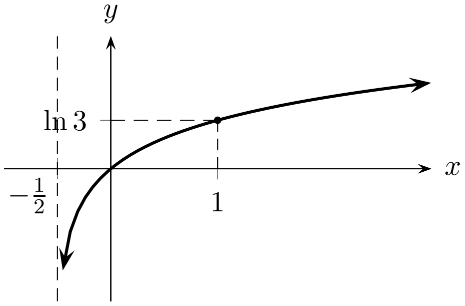
**1.** (A) **2.** (D) **3.** (B) **4.** (D) **5.** (C) **6.** (D)

### Question 7 (Lowe)

(a) (4 marks)

✓ [2] for graph.

✓ [1] each for domain & range.




$$D = \left\{ x : x \geq -\frac{1}{2} \right\} \qquad R = \{ y : y \in \mathbb{R} \}$$

(b) (2 marks)

✓ [1] for using log laws correctly.

✓ [1] for $\frac{2}{2x+1} - \frac{3}{3x-7}$

$$\begin{aligned} &\frac{d}{dx}\left(\log_e\left(\frac{2x+1}{3x-7}\right)\right) \\ &= \frac{d}{dx}\left(\log_e(2x+1) - \log_e(3x-7)\right) \\ &= \frac{2}{2x+1} - \frac{3}{3x-7} \left( = \frac{-17}{(2x+1)(3x-7)} \right) \end{aligned}$$

(c) (2 marks)

✓ [−1] for each incorrect step.

$$y = 5^x$$

Take log base $e$ on both sides,

$$\begin{aligned} \ln y = \ln 5^x &= x \ln 5 \\ \therefore y &= e^{x \ln 5} \\ \frac{d}{dx}\left(5^x\right) &= \frac{d}{dx}\left(e^{x \ln 5}\right) \\ &= (\ln 5) e^{x \ln 5} \\ &(= 5^x \ln 5) \end{aligned}$$

(d) (2 marks)

✓ [1] for correctly applying the product rule.

✓ [1] for correct simplification to $-3x^3e^{-3x} + 3x^2e^{-3x}$

$$\begin{aligned} y &= x^3 e^{-3x} \\ u = x^3 &\qquad v = e^{-3x} \\ u' = 3x^2 &\qquad v' = -3e^{-3x} \\ \frac{d}{dx}\left(x^3e^{-3x}\right) &= x^3\left(-3e^{-3x}\right) + 3x^2e^{-3x} \\ &= -3x^3e^{-3x} + 3x^2e^{-3x} \\ &\left(= 3x^2e^{-3x}(1-x)\right) \end{aligned}$$

### Question 8 (Ziaziaris)

(a) i. (2 marks)

✓ [−1] for each error.

$$\int \frac{x}{9+x^2}\,dx = \frac{1}{2}\log_e\left(9+x^2\right) + C$$

ii. ✓ [−1] for each error.

$$\int \frac{2}{x} + 5e^x\,dx = 2\log_e x + 5e^x + C$$

iii. ✓ [−1] for each error.

$$\begin{aligned} \int \frac{6-2x-x^2}{x}\,dx &= \int \frac{6}{x} - 2 - x\,dx \\ &= 6\log_e x - 2x - \frac{1}{2}x^2 + C \end{aligned}$$

iv. ✓ [−1] for each error.

$$\begin{aligned} \int x\left(x^2+4\right)^5\,dx &= \frac{1}{2}\int 2x\left(x^2+4\right)^5\,dx \\ &= \frac{1}{2} \times \frac{1}{6}\left(x^2+4\right)^6 + C \\ &= \frac{1}{12}\left(x^2+4\right)^6 + C \end{aligned}$$

(b) (2 marks)

- ✓ [−1] for each error.

$$\int_1^4 \frac{1}{x}\,dx = \Big[\log_e x\Big]_1^4$$
$$= \log_e 4 \quad (= 2\log_e 2)$$

**Question 9** (Lam)

(a) i. (2 marks)

- ✓ [1] for correctly applying the product rule.
- ✓ [1] for correct simplification.

$$y = x\ln x - x$$
$$u = x \quad v = \ln x$$
$$u' = 1 \quad v' = \frac{1}{x}$$
$$\frac{dy}{dx} = \cancel{x} \times \frac{1}{\cancel{x}} + \ln x - 1$$
$$= \ln x$$

ii. (2 marks)

- ✓ [1] for correct use of log law.
- ✓ [1] for correct answer.
- ✗ [0] for any other attempts to integrate $\ln(x^2)$.

$$\int \ln\left(x^2\right)\,dx = 2\int \ln x\,dx$$
$$= 2\left(x\ln x - x\right) + C$$
$$= 2x\ln x - 2x + C$$

(b) (4 marks)

- ✓ [1] for correct use of quotient rule.
- ✓ [1] for correct testing to discover horizontal point of inflexion.
- ✓ [1] for each correct coordinate.

$$y = \frac{e^x}{x^2+1}$$
$$u = e^x \quad v = x^2+1$$
$$u' = e^x \quad v' = 2x$$
$$\frac{dy}{dx} = \frac{e^x\left(x^2+1\right) - 2xe^x}{\left(x^2+1\right)^2}$$
$$= \frac{e^x\left(x^2-2x+1\right)}{\left(x^2+1\right)^2}$$
$$= \frac{e^x(x-1)^2}{\left(x^2+1\right)^2}$$

| $x$ | $1^-$ | $1$ | $1^+$ |
|---|---|---|---|
| $\frac{dy}{dx}$ | $+$ | $0$ | $+$ |
| $y$ | ↗ | $\left(1, \frac{1}{2}e\right)$ | ↗ |

Hence $\left(1, \frac{1}{2}e\right)$ is a horizontal point of inflexion.

**Question 10** (Weiss)

(a) (3 marks)

- ✓ [1] for correct primitive.
- ✓ [1] for correct substitution of point.
- ✓ [1] for final answer.

$$\frac{dy}{dx} = \frac{2}{2x-1}$$
$$y = \ln(2x-1) + C$$

When $x = 1$, $y = \ln 3$

$$\ln 3 = \ln(2-1) + C$$
$$\therefore C = \ln 3$$
$$\therefore y = \ln(2x-1) + \ln 3$$

(b) i. (2 marks)

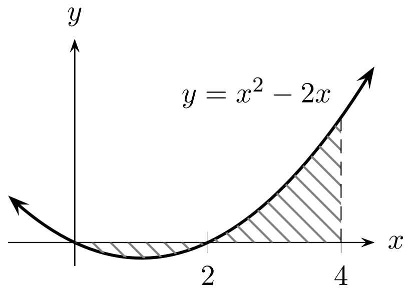
- ✓ [1] for correct primitive.
- ✓ [1] for final answer.

$$\int_0^4 x^2 - 2x\,dx = \left[\frac{1}{3}x^3 - x^2\right]_0^4$$
$$= \frac{64}{3} - 16 = \frac{16}{3}$$

ii. (2 marks)

✓ [1] for correct resolution into two separate integrals.

✓ [1] for correct final answer.




$$\begin{aligned} A &= \left|\int_0^2 x^2 - 2x\,dx\right| \\ &\quad + \int_0^4 x^2 - 2x\,dx \\ &= \left|\left[\frac{1}{3}x^3 - x^2\right]_0^2\right| + \left[\frac{1}{3}x^3 - x^2\right]_2^4 \\ &= \left|\frac{8}{3} - 4\right| + \frac{1}{3}\left(4^3 - 2^3\right) \\ &\quad - \left(4^2 - 2^2\right) \\ &= \frac{4}{3} + \frac{56}{3} - 12 \\ &= 8\,\text{units}^2 \end{aligned}$$

iii. (1 mark)
Part of the required area is below the $x$ axis.

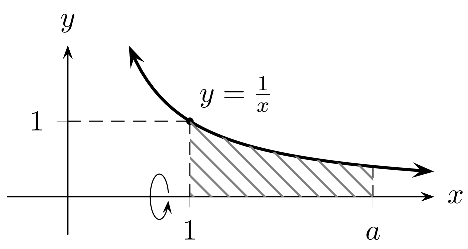
**Question 11** (Ireland)

(i) (3 marks)

✓ [1] for setting up correct integral.

✓ [1] for correct primitive.

✓ [1] for final answer.




$$\begin{aligned} V &= \pi \int_1^a \frac{1}{x^2}\,dx \\ &= \pi\left[-\frac{1}{x}\right]_1^a \\ &= \pi\left(1 - \frac{1}{a}\right) \ \text{units}^3 \end{aligned}$$

(ii) (1 mark)

$$\lim_{a\to\infty}\left[\pi\left(1 - \frac{1}{a}\right)\right] = \pi(1-0) = \pi$$

**Question 12** (Fletcher)

Let $P(n)$ be the statement

$$1\times2+3\times4+5\times6+\cdots+2n(2n-1) = \frac{n(n+1)(4n-1)}{3}$$

**Base case** $P(1)$:

$$\begin{gathered} 2(1)(2(1)-1) = 2(2-1) = 2 \\ \frac{1(1+1)(4(1)-1)}{3} = \frac{2\times3}{3} = 2 \end{gathered}$$

Hence $P(1)$ is true.

**Inductive hypothesis:** assume the $k$-th proposition $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e.

$$1\times2+3\times4+5\times6+\cdots+2k(2k-1) = \frac{k(k+1)(4k-1)}{3}$$

and examine $P(k+1)$:

$$\begin{gathered} 1\times2+3\times4+\cdots+2k(2k-1) \\ +\,2(k+1)(2(k+1)-1) \end{gathered}$$

$$\begin{aligned} &= \frac{k(k+1)(4k-1)}{3} + 2(k+1)(2k+1) \\ &= \frac{(k+1)\Big(k(4k-1)+6(2k+1)\Big)}{3} \\ &= \frac{(k+1)(4k^2-k+12k+6)}{3} \\ &= \frac{(k+1)(4k^2+11k+6)}{3} \\ &= \frac{(k+1)(4k+3)(k+2)}{3} \\ &= \frac{(k+1)((k+1)+1)(4(k+1)-1)}{3} \end{aligned}$$

$\therefore P(k+1)$ is also true. Hence $P(n)$ is true for integers $n \geq 1$ by induction.